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EARTH-MOON SYSTEM: DYNAMICS AND PARAMETER ESTIMATION

NUMERICAL CONSIDERATIONS AND PROGRAM DOCUMENTATION

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By

W.J. Breedlove, Jr.

Semiannual Progress Report
Covering the Period
August 17, 1975 to February 17, 1976

Prepared for the
National Aeronautics and Space Administration
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Hampton, Virginia

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I. INTRODUCTION

The period covered by this report has been spent primarily in coding and verifying the equations of motion for the Earth-Moon system as presented in Reference [1]. Some attention has also been given to numerical integration methods and parameter estimation methods. Existing analytical theories such as Brown's lunar theory as updated in [2]; Eckhardt's theory for lunar rotation, [3]; and Newcomb's theory for the rotation of the Earth, [4], have been coded and verified. These analytical theories serve as checks for the numerical integration. Laser ranging data for the period January 1969 - December 1975 has been collected and is stored on tape.

This report presents descriptions and verifications of the several programs developed to date. A discussion of the parameter estimation method to be used and certain supporting theoretical developments are also presented.

The main goal of this research is the development of software to enable physical parameters of the Earth-Moon system to be estimated making use of the extremely accurate data available from the Lunar Laser Ranging Experiment (LURE) and the Very Long Base Interferometry experiment (VLBI) of project Apollo.

A more specific goal is to develop software for the estimation of certain physical parameters of the Moon such as the inertia ratios α , β , γ , and the third and fourth harmonic gravity coefficients, C'_{3_j} , S'_{3_g} , C'_{4_b} , S'_{4_k} (j = 0, 1, 2, 3; k = 0, 1, 2, 3, 4). A unified model of the translational and rotational motion of the Moon is to be utilized in the estimation process. Also LURE data only will be processed.

In preparing the software, several basic questions must be asked, viz.,

A. Should existing software be used to model the dynamics of the Moon?

B. What type and order numerical integration scheme should be used?

C. How are the numerical integration results to be checked?

The answer to these questions--in part--is given below:

A. Existing software provides a piecemeal treatment of the problem in that three separate numerical integrations must be made:

i) Integration of all planets and Earth-Moon barycenter.

ii) Integration of Moon with respect to Earth-Moon barycenter.

iii) Integration of lunar rotational equations.

This is the approach used by the LURE Team and presented in the literature, [5]. This approach ignores the coupling of translational and rotational motions when terms of order $(1/r^4)$ in the mutual distance are retained in the differential equations. It also does not provide for the inclusion of mutual gravitational potential terms which arise when terms of $(1/r^5)$ are retained. Terms of at least the above orders must be retained to give physical libration accuracies of .005" to .01". This corresponds to a 2 cm to 3 cm accuracy in the LURE data.

In order to achieve a unified numerical model and to allow rigorous investigations of the coupling and mutual potential effects it was decided to develop new software. The dynamical model was discussed in [1] and is referred to as a unified model of lunar translation and rotation (UMLTR). It is a numerical integration of the combined lunar translational and rotational equations providing rigorously for coupling and mutual potential terms. This model is an integral part of two programs to be described later, viz., RIGEM and ESTEM.

B. The success of Oesterwinter and Cohen [6] in integrating the solar system using a high order "Cowell" type method led to the development of a numerical integration subroutine based on that method. In the process of developing and verifying this routine, a numerical integration developed by Everhart, [7], RAI9S was obtained. This is equivalent to a high order implicit Runge-Kutta scheme and is an implicit single-sequence method using Gauss-Radau and Gauss-Lobatto spacings. A new "Cowell" type routine started by RAI9S was then developed and is referred to as COW. Subroutine RAI9S has been used in all runs to date in the development of the dynamics although a comparison of RAI9S and COW is anticipated. The best integration order should also be decided based on a future study.

C. The practice in the literature has been to compare numerical models of lunar rotation with existing analytical theories. Eckhardt's theory [3] as modified by Williams [8] has been the basis for comparison. More recently, Eckhardt has developed a "300 Series" [9] lunar libration model.

To verify the unified model, comparisons must be made with both a lunar physical libration model and a solar system integration model. Initial conditions for the integration must be determined to obtain the best fit between the theory and the integration.

Program ANEAMO was developed to provide an evaluation of existing lunar libration theories for comparison purposes. The solar system motion to date has been compared with data found in Oesterwinter and Cohen's work, [6].

In summary, the programs developed or utilized to date and their capabilities are:

ANEAMO (01): This program evaluates 1) a truncated form of Brown's lunar theory, 2) Eckhardt's lunar physical libration theory, and 3) Newcomb's theory for the rotation of the Earth. It provides printed or punched output for use in other programs.

RAI9S: A single sequence integrator developed by Everhart, [7]. It integrates a system of N first or second order equations with orders of 7, 11, 15 or 19. Another similar routine RADAU31 has also been checked out which is capable of orders 7, 11, 15, 19, 23, 27, and 31. This latter routine requires double precision computations on the CDC 6600 series computers however.

COW: A multistep predictor-corrector method for the solution of first or second order systems of differential equations, [10]. The method is based on Bessel's central difference interpolation polynomials and the assumption that differences of some even order remain constant. The method is started with RAI9S. Orders of 4, 12, and 16 have been utilized. Starting is accomplished with a comparable order of RAI9S.

RIGEM (01): This program numerically integrates the combined translational and rotational motions of the Earth and Moon together with the translational motion of the remaining planets and Sun treated as particles. Options exist for integrating a subset of the planets and for eliminating Earth rotation. Subroutine RAI9S is utilized. Printed output or punched output is available.

ESTEM: The basic putpose of this program is to fit the numerically calculated values of the lunar libration angles using the UMLTR to those obtained from an analytical theory. The fit is made using an iterative weighted least squares approach with α , β , γ , and the initial Euler parameters and their rates being adjusted in the process. The basic dynamic model is RIGEM (01) (with Earth rotation integration removed). Necessary partial derivatives are generated using a numerical "variant trajectory" approach. The final residuals of the fit are output in plotted form.

This report includes documentation for ANEAMO (01), RIGEM (01), and ESTEM (01).

Finally a verification run of ESTEM is described which compares the UMLTR with Eckhardt's libration theory for one year. Final residuals are periodic with amplitudes of ±3" in the latitude librations $I\sigma$ and ρ and ±10" in the longitude libration, τ.

Work is in progress to reduce the size of those residuals to the 1" level.

Listings and decks for all programs and subroutines are available from the author.

II. PROGRAM DESCRIPTION AND VERIFICATION

A. Program ANEAMO (Version 01)

General. Program ANEAMO (01) provides the capability for evaluating analytical theories of the translational and rotational motions of the Earth and Moon. Version 01 provides the capabilities for

i) Evaluating a truncated form of Brown's lunar theory.

ii) Evaluating Eckhardt's theory for lunar physical librations.

iii) Evaluating Newcomb's expressions for the precession, nutation, and spin of the Earth,

iv) Providing orientation information in several useful forms such as direction cosines, Euler parameters, and the axis and angle of rotation, and

v) Providing punched card output of lunar physical librations in longitude and latitude.

Program and Subroutine Description. Salient features of the main program and subroutines including theoretical developments are presented here. Further information may be found in the program listings in Appendix A. Refer to Reference [1] for general theory.

Program ANEAMO. This program handles all input/output operations. It calls in succession the subroutines necessary to evaluate the various theories.

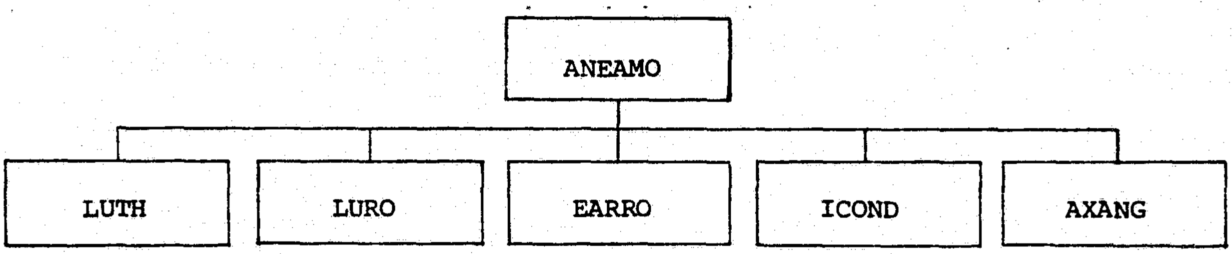
The program also provides for the calculation--by differentiation of a cubic spline fit--of the time derivatives of the Euler parameters, $\{\beta'_i\}$ and $\{\beta'''\}$ as evaluated from the rotational theories. These may then be used as initial guesses at the initial conditions for numerical integrations of the rotational equations of motion. The spline fit is accomplished using a standard subroutine SPLDER, [11].

This program evaluates all quantities at a series of Julian dates from some initial date VJIN to a final value VJF in steps of VJINC days.

A multi-case option may be exercised if the parameter ICODE = 1, otherwise if ICODE = 0 then the program stops.

Finally, the output parameter IPT may be set equal to 1 if it is desired to have the physical librations in longitude and latitude punched on cards. If this option is selected, a printout of these quantities is also made.

Program ANEAMO links several subroutines as follows:



Subroutine Luth. This subroutine contains a truncated form of Brown's lunar theory taken from Reference [2]. The fundamental arguments:

☾ , mean longitude of the Moon (V1),
Γ , Sun's mean longitude of perigee (V2),
Γ' , mean longitude of lunar perigee (V3),
Ω , longitude of mean ascending node of lunar orbit on the ecliptic (V4),
D , mean elongation of Moon from Sun (V5),
ε , mean obliquity of the ecliptic (V6),

were programmed as they appear in Reference [12]. They are equivalent to corresponding arguments L , $\tilde{\omega}'$, $\tilde{\omega}$, Ω , D , ε of the Improved Lunar Ephemeris, [2].

The Moon's longitude, latitude, and parallax with respect to the mean equinox and ecliptic of data are calculated from the formulae:

$$\text{Longitude} = L\ (i\theta) + \delta L\ (i\eta) + \delta L\ (i\alpha) + \delta L\ (i\delta) \tag{1}$$

$$S = F\ (i\theta) + \delta L\ (i\eta) - \delta\Omega\ (i\eta) + \delta S\ (i\beta) \tag{2}$$

$$\text{Latitude} = A \sin S + B \sin 3S + C \sin 5S + DN\ (i\beta) + \delta\ \text{Lat}\ (i\varepsilon) \tag{3}$$

$$\sin \Pi = (\delta \sin \Pi\ (i\gamma) + \delta \sin \Pi\ (i\zeta))\ (1 - 4.6747 \times 10^{-5}) \tag{4}$$

$$\text{Parallax} = \sin \Pi + (1/6 \sin^3\ (\Pi))/(206265)^2 \tag{5}$$

where

$L\ (i\theta) = ☾$

$F\ (i\theta) = ☾ - \Omega$

$\delta L\ (i\eta)$ = additive terms in longitude

$\delta L\ (i\alpha)$ = solar terms in longitude

$\delta L\ (i\delta)$ = planetary terms in longitude

$\delta\Omega\ (i\eta)$ = additive terms in node

$\delta S\ (i\beta)$ = solar terms in latitude

δ Lat $(i\varepsilon)$ = planetary terms in latitude

$N\ (i\beta)$ = solar terms in latitude

$A = \gamma_1 + \gamma_1 C$

$B = \gamma_2/\gamma_1 A$

$C = \gamma_3/\gamma_1 A$

$D = 1/\gamma_1 A$

$\gamma_1 C$ = solar terms in latitude

$\gamma_1, \gamma_2, \gamma_3$ = coefficients of principal terms in latitude.

The principal terms in latitude γ_1, γ_2, γ_3 were respectively multiplied by the first, third and fifth power of the factor $(1 + 2.708 \times 10^{-6} + 139.978\ \delta\gamma_C)$ where $\delta\gamma_C$ are additive terms from list $i\eta$.

All notations used above are explained in the ILE. Certain small correction factors have been ignored that are not consistent with the accuracy of the terms retained. Table 1 presents the accuracy of terms retained in version 01 of ANEAMO.

Table 1. Truncation Level of Brown's Lunar Theory.

Coordinate	Amplitude of smallest terms retained (rounded to value shown)
Longitude	1"
Latitude	1"
Parallax	.01"

Additive terms due to planetary perturbations and other small long period effects are included in ☾ , Ω , and $\delta\gamma_C$. Table 2 lists the terms programmed in version 01 of ANEAMO.

Table 2. Additive Terms Programmed in ANEAMO.

Ref. No.	Add To	Coefficient	Planet
1365	L	0.84	--
1366	L	0.31	Venus
1373	L	14.27	Venus
1375	L	7.261	--
1376	L	0.282	--
1379	L	0.075	Mercury
1382	L	0.237	Venus
1383	L	0.108	Venus
1385	L	0.126	Venus
1369	Ω	0.63	--
1406	Ω	0.17	Venus
1407	Ω	95.96	--
1408	Ω	15.58	--
1409	Ω	1.86	--
1413	$\delta\gamma_C$	-4.318	--
1414	$\delta\gamma_C$	-0.698	--
1415	$\delta\gamma_C$	-0.083	--

This subroutine also calculates the geocentric ecliptive coordinates $\{x_i\}_{ECL}$ and geocentric equatorial coordinates $\{x_i\}_{EQ}$ of the Moon. These are mean equinox and ecliptic of date and mean equator and equinox of date systems respectively. Formulae utilized for this computation are

$$r = (6378.16)/\text{Parallax} \tag{6}$$

$$\{x_i\}_{ECL} = \begin{Bmatrix} r \cos(\text{latitude}) \cos(\text{longitude}) \\ r \cos(\text{latitude}) \sin(\text{longitude}) \\ r \sin(\text{latitude}) \end{Bmatrix} \tag{7}$$

$$\{x_i\}_{EQ} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{bmatrix} \{x_i\}_{ECL} . \tag{8}$$

Subroutine LURO. This subroutine calculates the physical librations in longitude, node, and inclination based on Eckhardt's librational theory contained in References [3] and [13]. The additive and planetary terms constructed by Williams in Reference [8] are also included. This is basically a second degree theory in the lunar gravity harmonic coefficients. Eckhardt's theory provides the physical librations in the form

$$\tau_j = \sum_i T_{ij} \sin(NL_i \cdot \ell + NLP_i \cdot \ell' + NF_i \cdot F + ND_i \cdot D)$$

$$I\sigma_j = \sum_k C_{kj} \sin(NL_k \cdot \ell + NLP_k \cdot \ell' + NF_k \cdot F + ND_k \cdot D) \tag{9}$$

$$\rho_j = \sum_m R_{mj} \sin(NL_m \cdot \ell + NLP_m \cdot \ell' + NF_m \cdot F + ND_m \cdot D)$$

where

τ_j = physical libration in longitude

$I\sigma_j$ = physical libration in latitude

ρ_j = physical libration in inclination

I = inclination of lunar equator to ecliptic

j = index number for set of coefficients being used

i, k, m = summation indices

$\ell = L - \tilde{\omega}$

$\ell' = L' - \tilde{\omega}'$

$F = L - \Omega$

$D = L - L'$

NL, NLP, NF, ND = integer coefficients

The coefficients T_{ij} , C_{kj} , R_{wj} depend on the inertia ratios β , γ of the Moon, where

$$\beta = (C - A)/B$$

$$\gamma = (B - A)/C$$

and A , B , C are the lunar principle inertias. Thus, the j index distinguishes between the coefficients for different values of β and γ . Currently coefficients for three j's have been used as shown in Table 3 (see Appendix A for numerical values).

Table 3. Eckhardt Coefficients Programmed.

j	I	β	γ
1	5521.6"	6.268×10^{-4}	2.3×10^{-4}
2	5559.6"	6.3×10^{-4}	2.0×10^{-4}
3	5550.2"	6.293×10^{-4}	2.27×10^{-4}

The additive and planetary contributions as calculated by Williams in [8] are for $\beta = 6.293 \times 10^{-4}$ and $\gamma = 2.27 \times 10^{-4}$. The $j = 1$ coefficients come directly from one of Eckhardt's tables and are rounded to 1". The coefficients in $j = 2$ and $j = 3$ were obtained by interpolating between various tables given by Eckhardt keeping terms to 0.2" in one of the tables interpolated.

The inclination of the lunar equator to the ecliptic, I, is also a function of β and γ. The values used for I are also shown in Table 3 as calculated from Eckhardt's empirical formula

$$I = -1612'' - 5.2 \times 10^4 \gamma + 11.4 \times 10^6 \beta . \tag{10}$$

Once the physical librations are calculated, the Moon's orientation in inertial space is calculated as follows.

The physical librations τ, $I\sigma$, ρ are perturbations to three Euler angles ψ, θ, $\tilde{\phi}$ locating the Moon with respect to the ecliptic and mean equinox of date. To express this mathematically, consider the geometry of Figure 1. The $\{z_i\}$ axes are fixed to the rigid Moon and are principle center-of-mass axes. The axes $\{x_i\}_{ECL}$ are the ecliptic and mean equinox of date set. The axis notation used here follows that defined in Reference [1].

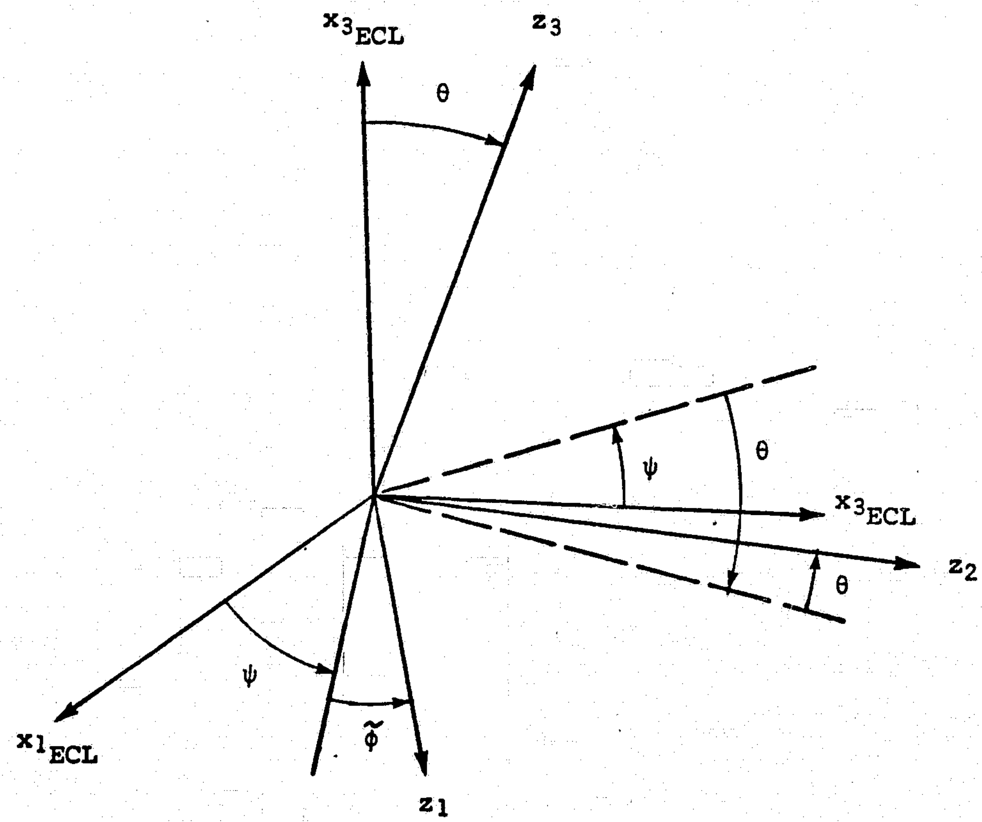


Figure 1. Orientation of Moon in Space.

The above axes are related as follows:

$$\{z_i\} = [RM]\ \{x_i\}_{ECL} \tag{11}$$

where

$$[RM] = \begin{bmatrix} (C\psi\ C\tilde{\phi} - S\tilde{\phi}\ S\psi\ C\theta) & (C\tilde{\phi}\ S\psi + S\tilde{\phi}\ C\theta\ C\psi) & -S\tilde{\phi}\ S\theta \\ (-C\psi\ S\tilde{\phi} - C\tilde{\phi}\ C\theta\ S\psi) & (-S\tilde{\phi}\ S\psi + C\tilde{\phi}\ C\psi\ C\theta) & -C\tilde{\phi}\ S\theta \\ -S\psi\ S\theta & C\psi\ S\theta & C\theta \end{bmatrix}$$

The matrix [RM] is a rotation matrix for a rotation from $\{x_i\}_{ECL}$ to $\{z_i\}$ in the sense ZXZ through angles ψ , $-\theta$, $\tilde{\phi}$. The shorthand notation $S\tilde{\phi} \equiv \sin\tilde{\phi}$ and $C\tilde{\phi} = \cos\tilde{\phi}$ is used above.

Now, the Euler angles and physical librations are related through the expressions

$$\left.\begin{aligned} \theta &= I + \rho \\ \psi &= \Omega + \sigma \\ \tilde{\phi} &= \Pi + ☾ - \psi + \tau \end{aligned}\right\} , \tag{12}$$

where I = constant and Ω and ☾ are the fundamental arguments from the lunar theory.

The orientation of the Moon with respect to the mean equator and equinox of 1950.0 frame, $\{X_i'\}$, may be obtained as follows:

$$\{z_i\} = [RM]\ [EC]^T\ [P]\ \{x_i'\} \tag{13}$$

where

$$\{x_i\}_{EQ} = [EC]\ \{x_i\}_{ECL}$$

$$\{x_i\}_{EQ} = [P]\ \{x_i'\}$$

[P] = Precession matrix (defined later)

$$[EC] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\varepsilon & -S\varepsilon \\ 0 & SG & C\varepsilon \end{bmatrix} .$$

The orientation of the Moon with respect to the reference axes $\{Z_i\}$ for the Moon may be obtained as follows:

$$\{z_i\} = [XM\phi]\ \{Z_i\} . \tag{14}$$

In order to evaluate matrix $[XM\phi]$ the relation between $\{X_i'\}$ and $\{Z_i\}$ must be known. This depends only on the position of the Moon's center of mass with respect to $\{X_i'\}$ as discussed in Figure 2 of Reference [1]. Referring to that figure, the relation between $\{X_i'\}$ and $\{Z_i\}$ can be obtained by

i) Rotating $\{X_i'\}$ through $(\lambda + \Pi)$ about X_3' to obtain $\{Z_i'\}$ and then

ii) Rotating $\{Z_i'\}$ through ϕ about Z_2' to obtain $\{Z_i\}$.

The compound rotation provides

$$\{Z_i\} = [T]\ \{X_i'\} \tag{15}$$

where

$$[T] = \begin{bmatrix} -C\lambda\ C\phi & -C\phi\ S\lambda & -S\phi \\ S\lambda & -C\lambda & 0 \\ -C\lambda\ S\phi & -S\lambda\ S\phi & C\phi \end{bmatrix} .$$

Here ϕ and λ are the geocentric latitude and longitude of the Moon with respect to the mean equator and equinox of 1950.0. The Moon's position, $\{X_i\}_{EQ50}$ is obtained from Brown's lunar theory as calculated in LUTH. Thus,

$$\{x_i\}_{EQ50} = [P]^T\ \{x_i\}_{EQ} .$$

Next, the polar coordinates r , ϕ , λ as required in (15) are found from

$$
\left.\begin{aligned}
r &= \sqrt{x_{1EQ50}^2 + x_{2EQ50}^2 + x_{3EQ50}^2} \\
C\phi\ C\lambda &= x_{1_{EQ50}}/r \\
C\phi\ C\lambda &= x_{2_{EQ50}}/r \\
S\phi &= x_{3_{EQ50}}/r \\
C\phi &= \sqrt{x_{1EQ50}^2 + x_{2EQ50}^2}\ /r \\
S\lambda &= C\phi\ S\lambda/C\phi \\
C\lambda &= C\phi\ C\lambda/C\phi\ .
\end{aligned}\right\} \tag{16}
$$

Finally, the matrix $XM\phi$ is constructed as follows

$$
[XM\phi] = [RM]\ [EC]^T\ [P]\ [T]^T\ , \tag{17}
$$

thus providing the orientation of the Moon with respect to the $\{Z_i\}$ reference axes.

LURO also calculates the Earth's selenographic coordinates, λ_e and μ_e .

By definition, the Earth's selenographic coordinates are the latitude (μ_e) and longitude (λ_e) of the Earth as seen from the $\{z_i\}$ axes. Thus,

$$
\left.\begin{aligned}
C\mu_e\ C\lambda_e &= -\vec{k}_1 \cdot \vec{i}_r = +\ \vec{k}_1 \cdot \vec{K}_1 \\
C\mu_e\ S\lambda_e &= -\vec{k}_2 \cdot \vec{i}_r = \vec{k}_2 \cdot \vec{K}_2 \\
S\mu_e &= -k_3 \cdot \vec{i}_r = \vec{k}_3 \cdot \vec{K}_3
\end{aligned}\right\} \tag{18}
$$

where $\vec{i}_r$ is a unit vector directed from the Earth's mass center to the lunar mass center as shown in Figure 2 of Reference [1]. Since $\vec{i}_r = -\vec{K}_1$, the direction cosines in (18) are just the elements of the first column of matrix $XM\phi$. Then

$$
\left.\begin{aligned}
\lambda_e &= \tan^{-1} \frac{XM\phi\ (2,1)}{XM\phi\ (1,1)} \\
\mu_e &= \tan^{-1} \frac{XM\phi\ (3,1)}{\sqrt{XM\phi\ (1,1)^2 + XM\phi\ (2,1)^2}}
\end{aligned}\right\} \tag{19}
$$

Subroutine EARRO. Version 01 of EARRO contains the approximate expressions for Earth rotation provided in Reference [4].

The true sidereal time, accurate to 0.2" or 10^{-6} is given by

$$\begin{aligned}\theta = {} & 100.075542 + 360.985647348\ T \\ & + 0.29 \times 10^{-12}\ T^2 - 4.392 \times 10^{-3} \sin(a_1) \\ & + 0.053 \times 10^{-3} \sin(a_2) - 0.325 \times 10^{-3} \sin a_3 \\ & - 0.05 \times 10^{-3} \sin(a_4)\end{aligned} \tag{20}$$

where

$$\begin{aligned} T &= \text{Julian Date} - 2433282.5 \\ a_1 &= 12^\circ\!.1128 - 0.052954\ T \\ a_2 &= 2a_1 \\ a_3 &= 2\ (280^\circ\!.0812 + 0.985647\ T) \\ a_4 &= 2\ (64^\circ\!.3824 + 13.176398\ T)\ . \end{aligned}$$

The transformation from the $\{x_i'\}$ system to the $\{y_i\}$ system is given in Reference [4] as

$$\{y_i\} = [S]\ [N]\ [P]\ \{x_i'\} \tag{21}$$

where [S] is the spin matrix, [N] is the nutation matrix, and [P] is the precession matrix.

More explicitly,

$$[S] = \begin{bmatrix} C\theta & S\theta & 0 \\ -S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{22}$$

$$[P] = \left[\begin{array}{c|c|c} (-S\kappa\ S\omega + C\kappa\ C\omega\ C\nu) & (-C\kappa\ S\omega - S\kappa\ C\omega\ C\nu) & -C\omega\ S\nu \\ (S\kappa\ C\omega + C\kappa\ S\omega\ C\nu) & (C\kappa\ C\omega - S\kappa\ S\omega\ C\nu) & -S\omega\ S\nu \\ C\kappa\ S\nu & -S\kappa\ S\nu & C\nu \end{array}\right], \tag{23}$$

and

$$[N] = \begin{bmatrix} C\ (\Delta\nu)\ C\ (-\Delta\mu) & C\ (\Delta\nu)\ S\ (-\Delta\mu) & -\ S\ (\Delta\nu) \\ (C\ (-\Delta\mu)\ S\ (-\Delta\varepsilon)\ S\ (\Delta\nu) & (S\ (-\Delta\mu)\ S\ (-\Delta\varepsilon)\ S\ (\Delta\nu) & S\ (-\Delta\varepsilon) \\ \quad -\ S\ (-\Delta\mu)\ C\ (-\Delta\varepsilon)) & \quad +\ C\ (-\Delta\varepsilon)\ C\ (-\Delta\mu)) & \quad C\ (\Delta\nu) \\ (C\ (-\Delta\mu)\ S\ (\Delta\nu)\ C\ (-\Delta\varepsilon) & (S\ (-\Delta\mu)\ S\ (\Delta\nu)\ C\ (-\Delta\varepsilon) & C\ (-\Delta\varepsilon) \\ \quad +\ S\ (-\Delta\varepsilon)\ S\ (-\Delta\mu)) & \quad -\ S\ (-\Delta\varepsilon)\ C\ (-\Delta\mu)) & \quad C\ (\Delta\nu) \end{bmatrix} \quad (24)$$

The rigorous form of the nutation matrix rather than the approximate form

$$[N] = \begin{bmatrix} 1 & -\Delta\mu & -\Delta\nu \\ \Delta\mu & 1 & -\Delta\varepsilon \\ \Delta\nu & \Delta\varepsilon & 1 \end{bmatrix} \quad (25)$$

was programmed to insure that [N] was rigorously orthonormal.

The arguments utilized in the above matrices are

$$\begin{aligned} \kappa &= 0^\circ\!.063107\ T \\ \omega &= 0^\circ\!.063107\ T \\ \nu &= 0^\circ\!.0548757\ T \end{aligned} \quad (26)$$

$$\begin{aligned} \Delta\mu &= -76.7 \times 10^{-6} \sin(a_1) + 0.9 \times 10^{-6} \sin(a_2) \\ &\quad -5.7 \times 10^{-6} \sin(a_3) - 0.9 \times 10^{-6} \sin(a_4) \\ \Delta\nu &= -33.3 \times 10^{-6} \sin(a_1) + 0.4 \times 10^{-6} \sin(a_2) \\ &\quad -2.5 \times 10^{-6} \sin(a_3) - 0.4 \times 10^{-6} \sin(a_4) \\ \Delta\varepsilon &= 44.7 \times 10^{-6} \cos(a_2) - 0.4 \times 10^{-6} \cos(a_2) \\ &\quad + 2.7 \times 10^{-6} \cos(a_3) + 0.4 \times 10^{-6} \cos(a_4) \;. \end{aligned} \quad (27)$$

Note that $\Delta\mu$, $\Delta\nu$, $\Delta\varepsilon$ are given in radians.

Reference [4] lists the accuracy of the above expressions as 0.2" or 10^{-6} .

The values given in equations (26) and (27) are programmed. If more accurate values are required, the following may be used.

$$\kappa = (23042''53 + 139''73\ \tau + 0''06\ \tau^2)\ t$$

$$+ (30''23 - 0''27\ \tau)\ t^2 + 18''00\ t^3$$

$$\omega = \kappa + 179''27 + 0''66\ \tau)\ t^2 + 0''32\ t^3 \tag{28}$$

$$\nu = (20046''85 - 85''33\ \tau - 0''37\ \tau^2)\ t$$

$$+ (-42''67 - 0''37\ \tau)\ t^2 - 41.80\ t^3$$

where τ is the epoch (1950.0) of $\{X_i'\}$ and t is the epoch of the mean sidereal system, both measured in thousands of tropical years from 1900.0. Also many additional nutation terms are listed in References [2] and [14].

Subroutine ICOND. This subroutine accomplishes a transformation from the Euler parameters $\{\beta''\}$ for the Earth to the Euler parameters $\{\beta'\}$. The relation between those was presented in equation (16) of Reference [1] and is

$$\{\beta'\} = [\beta]^{-1}\ \{\beta''\}\ .$$

The Euler parameters $\{\beta\}$ locate the reference axis system $\{Y_i\}$ with respect to the system $\{X_i'\}$. Since the $\{Y_i\}$ system is rotated, through an angle, α, about the $\overline{I}_3'$ axis with respect to $\{X_i'\}$ can be shown that

$$\begin{aligned} \beta_0 &= \cos\ (\alpha/2) \\ \beta_1 &= 0 \\ \beta_2 &= 0 \\ \beta_3 &= \sin\ (\alpha/2)\ , \end{aligned} \tag{29}$$

where

$$\alpha = \alpha_0 + \dot{\alpha}\ T. \tag{30}$$

Thus,

$$[\beta] = \begin{bmatrix} C\ (\alpha/2) & 0 & 0 & -S\ (\alpha/2) \\ 0 & C\ (\alpha/2) & -S\ (\alpha/2) & 0 \\ 0 & S\ (\alpha/2) & C\ (\alpha/2) & 0 \\ S\ (\alpha/2) & 0 & 0 & C\ (\alpha/2) \end{bmatrix}\ . \tag{31}$$

Due to the nature of $[\beta]$

$$[\beta]^{-1} = [\beta]^T . \tag{32}$$

Currently,

$$\alpha_0 = 100.075542$$

$$\dot{\alpha} = 360.985647348 ;$$

are programmed.

Subroutine AXANG. This subroutine computes the axis and angle of rotation from any rotation matrix $[R]$ as well the corresponding Euler parameters $\{\beta_i\}$. The sense of the rotation is from $\{x\}$ to $\{x'\}$ where

$$\{x'\} = [R]\ \{x\} . \tag{33}$$

The formulae for this are provided in Reference [15] and are summarized below. The angle of rotation is δ and the direction cosines of the axis--with respect to both $\{x\}$ and $\{x'\}$ --are $\{C_i\}$.

$$\cos \delta = 1/2\ (R_{11} + R_{22} + R_{33} - 1) \tag{34}$$

$$\left.\begin{aligned} C_1 &= (a_{23} - a_{32})/2 \sin \delta \\ C_2 &= (a_{31} - a_{13})/2 \sin \delta \\ C_3 &= (a_{12} - a_{21})/2 \sin \delta \end{aligned}\right\} . \tag{35}$$

The Euler parameters for the rotation may be found from

$$\beta_0 = \cos\ (\delta/2) \tag{36}$$

$$\beta_i = C_i \sin\ (\delta/2) \qquad (i = 1, 2, 3) .$$

Logic is incorporated in this subroutine that keeps the calculated rotation axis generally aligned with the body rotation axis.

Verification of ANEAMO. The verification of program ANEAMO has been accomplished in several ways

i) Table 4 provides a comparison of the calculated values of the fundamental arguments of the lunar theory as calculated by ANEAMO and as given in the American Ephemeris and Nautical Almanac (AENA) for 1974, [12].

ii) Table 5 provides the residuals in geocentric ecliptic longitude, latitude and parallax from a comparison of ANEAMO with the AENA. The sense of the residuals is ANEAMO-AENA. Note that the nutation in longitude must be added to the values calculated in ANEAMO for the comparison. The calculations are made for a two-month period beginning at J. D. 2442050.5. Table 5 shows actual residuals, residuals, their mean $(\bar{x})$ and standard deviation (SD) for the case when the longitude and latitude series are truncated at 1" and the parallax series is truncated at $0\overset{''}{.}01$. Additional terms in the series are to be programmed but the accuracy shown in Table 5 is sufficient for present purposes.

iii) The sidereal time, θ, as calculated in EARRO was compared with the value given in the AENA for J. D. 2442050.5. A residual of $0\overset{''}{.}3$ resulted.

iv) The calculation of the nutation matrix elements $\Delta\nu$, $-\Delta\varepsilon$, and $-\Delta\mu$ was compared with values presented in the AENA. For J. D. 2442050.5, ANEAMO provides.

$$\Delta\mu = 0.78227 \times 10^{-4} \text{ rad.}$$

$$\Delta\nu = 0.33966 \times 10^{-4} \text{ rad.}$$

$$\Delta\varepsilon = -0.36289 \times 10^{-5} \text{ rad.} = -0.748''$$

Comparable values to the above are not presented in the AENA but $\Delta\psi$ and $\Delta\varepsilon$ are given there. The values are

$$\Delta\varepsilon = -0\overset{''}{.}740$$

$$\Delta\psi = 17\overset{''}{.}493 \; .$$

The nutations $\Delta\mu$ and $\Delta\nu$ are related to $\Delta\psi$ through

$$\Delta\psi = \Delta\mu/\cos\varepsilon = \Delta\nu/\sin\varepsilon$$

Table 4. Verification of Fundamental Argument Calculation in ANEAMO

Quantity	ANEAMO††	AENA††
	35°02006	35°0201
Γ	282°49337	282°49335*
Γ'	105°63199	105°6320
Ω	267°81342	267°8134
D	112°78291	112°7829
ε	23°44266	23°44266†
ℓ	289°38807	289°3881
L	282°23714	282°2371
ℓ'	359°74377	359°74375
F	127°20664	127°2067

†† All comparisons made for J. D. 2442050.5.

* Hand calculation.

† Tabular value of ε in AENA is an "of date" value. This is related to the mean value, $\bar{\varepsilon}$, as calculated in ANEAMO by $\varepsilon_{0_D} = \bar{\varepsilon}$ + Nutation in obliquity.

Table 5. Residuals in Longitude, Latitude, and Parallax Between ANEAMO and AENA.

Day	Residuals in Longitude (")	Residuals in Latitude (")	Residuals in Parallax (")
0	-4.03	6.19	-.108
5	-1.19	-1.12	-.18
10	-3.38	-8.46	-.108
15	-7.24	-0.04	-.108
20	-5.51	-0.68	-.108
25	-4.75	-5.08	0
30	-1.62	-2.23	-.18
35	-4.82	4.32	-.108
40	-1.33	0.68	-.072
45	-3.67	1.22	-.072
50	-1.22	-3.13	-.036
55	.04	-6.52	.108
60	8.46	0.83	.108
$\bar{x}$	-2.33	-1.85	-.08
SD	3.85	3.78	.08

thus the residuals in $\Delta\varepsilon$ and $\Delta\psi$ may be calculated as

$$\text{Res. in } \Delta\psi = 0''\!.1$$

$$\text{Res. in } \Delta\varepsilon = 0''\!.008 \ .$$

v) The precession matrix [P] as calculated in ANEAMO for the date 1974.0 (J. D. 2442048.2358) could be compared with the same matrix appearing in the AENA. Out of the nine elements the worst case residual was 4.76×10^{-7}.

vi) The verification of the calculation of ρ, $I\sigma$, and $\dot{\tau}$, the physical librations, is manifested by a comparison of the theoretical values computed in ANEAMO with numerically integrated values. This comparison will be discussed later.

vii) Any set of Euler parameters must satisfy the following constraints:

$$\beta_0^2 + \beta_1^2 + \beta_2^2 + \beta_3^2 = 1 \tag{37}$$

$$\beta_0\,\dot{\beta}_0 + \beta_1\,\dot{\beta}_1 + \beta_2\,\dot{\beta}_2 + \beta_3\,\dot{\beta}_3 = 0 \tag{38}$$

These constraints are tested in program ANEAMO. The constraint (37) is satisfied more accurately than (38) since in the latter the rates $\dot{\beta}_i$ are calculated from a cubic spline fit to the parameters β_i. Typical values encountered are

Moon:

$$\Sigma\,\beta_i^2 - 1 = 1.8 \times 10^{-14}$$

$$\Sigma\,\beta_i\,\dot{\beta}_i = -7.5 \times 10^{-13}$$

Earth:

$$\Sigma\,\beta_i^2 - 1 = 3.2 \times 10^{-14}$$

$$\Sigma\,\beta_i\,\dot{\beta}_i = 2.2 \times 10^{-8}$$

B. Subroutines RAI9S, RADAU31

These are numerical integration routines for systems of first or second order ordinary differential equations. The theoretical development of the method used is presented in Reference [7]. Basically the solutions to $\dot{x} = F(x, t)$ are developed in truncated series in time t whose coefficients are found empirically. The method is a single-sequence method that uses Gauss-Radau and Gauss-Lobatto spacings for the several substeps within each sequence. The method is equivalent in principle with the implicit Runge-Kutta methods.

Subroutine RAI9S. This is a single precision deck suitable for a computer with a 60 bit word length in that precision. Integration orders of 7, 11, 15, and 19 are provided. This routine is used both in programs RIGEM and ESTEM and as a starter to the Cowell second-sun method of subroutine COW.

Subroutine RADAU31. This is a double precision deck suitable for CDC machines with a 120 bit word or for IBM machines with a 128 bit word. Integration orders of 7, 11, 15, 19, 23, 27, and 31 are provided.

Verification. Before use both of the above subroutines were verified by the calculation of a test orbit from the restricted three body problem, [10]. The orbit is shown in Figure 2. It is a periodic orbit in the rotating frame with period

$$t_f = 6.1921693313\ 19639\ 70699\ 23217\ \ldots\ .$$

Initial conditions are

$$y_1 = 1.2 \qquad y_1' = 0.0$$

$$y_2 = 0.0$$

$$y_2' = -1.04935\ 75098\ 3031990731\ 0410434\ \ldots\ .$$

The orbit has three loops and requires frequent step size changes.

Table 6 presents the results of the verification.

C. Subroutine COW.

The success of Oesterwinter and Cohen in integrating large systems of equations using high-order multistep methods [6] led to the development of this subroutine. The method was programmed in accordance with Reference [10]. It is referred to as Cowell's method and is a multistep predictor-corrector routine for systems of first or second order ordinary differential equations. Cowell's method

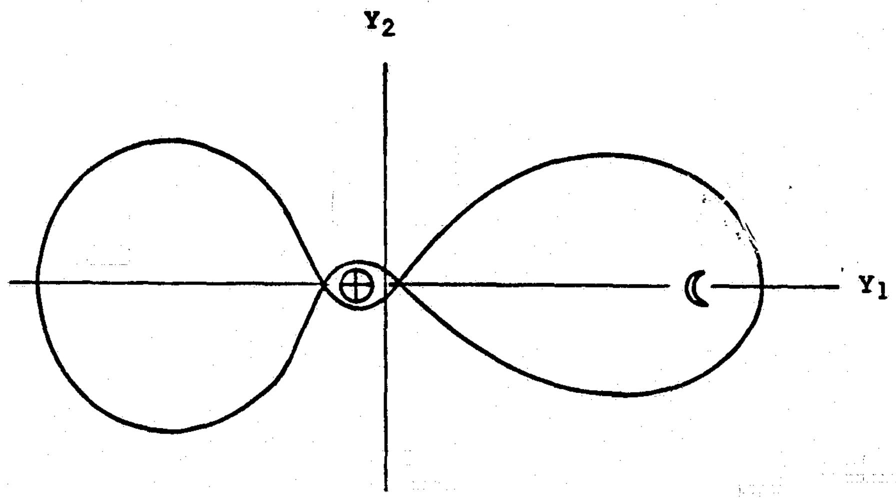


Figure 2. Test Orbit for Subroutines RA19S and RADAU31 from Restricted 3-Body Problem. (⊕ = Earth, ☾ = Moon)

Table 6. Accuracy of RAI9S and RADAU31 on test orbit.

Method	Order	Δy_1	Δy_1	Function Calls
RAI9S	7 (5)*	1.7×10^{-5}	3.4×10^{-5}	736
RAI9S	11 (6)	1.1×10^{-7}	1.9×10^{-7}	1025
RAI9S	11 (7)	4.6×10^{-9}	7.0×10^{-9}	1437
RAI9S	15 (10)	4.5×10^{-13}	4.9×10^{-13}	2867
RAI9S	19 (12)	2.3×10^{-15}	2.6×10^{-15}	3802
RADAU31	23 (15)	2.1×10^{-21}	3.6×10^{-19}	5820
RADAU31	27 (20)	1.0×10^{-24}	1.6×10^{-24}	39610874

* Numbers in parentheses are sequence size control numbers.

is based on Bessel's central difference polynomials and the assumption that differences of some even order remain constant, [10]. The coefficients for 4th, 12th, and 16th order methods have been punched. The method is started using RAI9S of a comparable order. A constant or variable step capability as well as an "exact end" capability is included. The "exact end" capability uses the RAI9S routine also. Options exist for no corrector, n applications of the corrector or iteration using the corrector until convergence is obtained within a specified accuracy level.

Verification. The same test orbit utilized for RAI9S has been integrated using COW. The results are shown in Table 7.

Table 7. Verification of COW using test orbit.

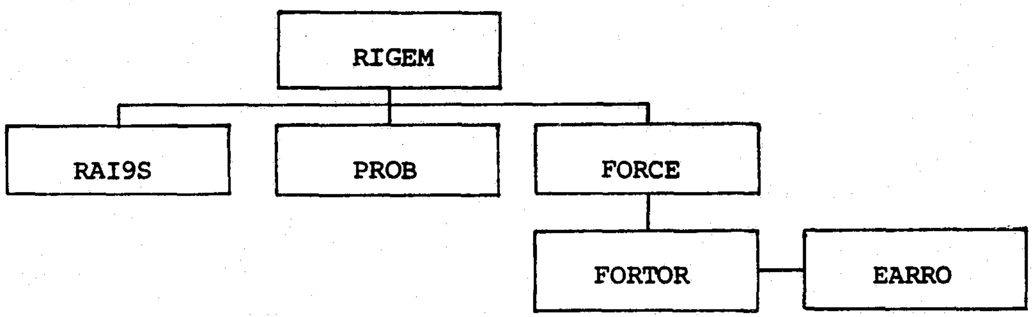
Order		Halving/Doubling Limit	Δy_1	$\Delta \dot{y}_1$	Function Calls
Start	Run				
11	12	$1. \times 10^{-10}$	2.0×10^{-7}	3.3×10^{-10}	8265*
11	12	$1. \times 10^{-11}$	2.2×10^{-10}	1.4×10^{-10}	10180
11	12	$5. \times 10^{-12}$	3.8×10^{-9}	1.2×10^{-8}	10761

* Iteration using corrector formulae produced large number of function calls.

The orbit integrated here places a stringent test on COW since step size changes are clumsy to handle with these types of methods. More work needs to be done on this subroutine in terms of optimization and criterion for choice of order and other parameters.

D. Program RIGEM (Version 01)

This program provides the capability for numerically integrating the coupled translational/rotational motions of the Earth and Moon treated as arbitrary rigid bodies together with the remaining planets and Sun modeled as particles. The general theory is outlined in Reference [1]. Subroutine RAI9S is utilized as the integration routine. Options exist for 1) omitting Earth rotation; 2) omitting planets Mercury, Saturn, Uranus, Neptune and Pluto (ϕMPL (1) = 1); 3) for multiple cases; and 4) for punched output of the calculated values of the lunar physical librations. The main program links several subroutines as follows



<u>Program RIGEM</u>. This program calls subroutine PROB to obtain all initial conditions and values of parameters. It then provides an integration loop that calls subroutine RAI9S at TINC intervals from VJDEP to VJDEP + TMAX. The loop also provides for output of calculated quantities and for calculation of the Earth's selenographic coordinates, the lunar physical librations and Euler parameter tests for Earth and Moon orientation.

At each time step subroutine RAI9S returns the following current values to RIGEM:

A. Current time TWR = VJDEP + TF .

B. Position and velocity of planets and Sun with respect to the $\{x_i'\}$ frame.

C. Euler parameters and rates $\{\beta_i'\}$, $\{\dot{\beta}_i'\}$, $\{\beta_i'''\}$, $\{\dot{\beta}_i'''\}$; as listed in Table 8.

This program calculates the Euler parameter tests for the Earth mentioned earlier

$$\Sigma\ \beta_i' = 1$$

$$\Sigma\ \beta_i'\ \dot{\beta}_i' = 0\ .$$

Since the immediate use of this program is in analyzing lunar motion, the lunar rotation segment has been more thoroughly treated than that of the Earth. Checks for gross errors have been made in the Earth rotation logic but no definitive verification studies have been made to date. Currently the Earth orientation is provided by subroutine EARRO as discussed earlier.

Table 8. Integration variables in RIGEM.

Translation			
Planet	Mass	Position	Velocity
Sun	1	x (1 - 3)	v (1 - 3)
Mercury	2	x (4 - 6)	v (4 - 6)
Venus	3	x (7 - 9)	v (7 - 9)
Earth	4	x (10 - 12)	v (10 - 12)
Moon	5	x (13 - 15)	v (13 - 15)
Mars	6	x (16 - 18)	v (16 - 18)
Jupiter	7	x (19 - 21)	v (19 - 21)
Saturn	8	x (22 - 24)	v (22 - 24)
Uranus	9	x (25 - 27)	v (25 - 27)
Neptune	10	x (28 - 30)	v (28 - 30)
Pluto	11	x (31 - 33)	v (31 - 33)

Rotation		
	Parameter	Rates
Earth $\{\beta_i'\}$, $\{\dot{\beta}_i'\}$	x (34 - 37)	v (34 - 37)
Moon $\{\beta_i'''\}$ $\{\dot{\beta}_i'''\}$	x (38 - 41)	v (38 - 41)

The rotation matrix $[C(\beta''')]$ defined by

$$\{z_i\} = [C(\beta''')]\{Z_i\} \tag{39}$$

can be evaluated at each step using $\{\beta_i'''\}$. The matrix $[C(\beta''')]$ is of the form given in equation (72) of Reference [1]. As shown earlier the Earth's selenographic longitude and latitude, λ_e and μ_e, can be calculated from the elements of the first column of $[C(\beta''')]$ as done in equation (15).

Next, the location of the Moon with respect to the Earth componentiated in the $\{X_i'\}$ frame can be found as follows:

$$\left.\begin{aligned} DAL(1) &= x(13) - x(10) \\ DAL(2) &= x(14) - x(11) \\ DAL(3) &= x(15) - x(12) \end{aligned}\right\} \tag{40}$$

The polar coordinates r, λ, ϕ may be found from equations (16) substituting DAL (1) for $x1_{EQ50}$, etc. Next, matrix $[T]$ of equation (15) may be formed.

The physical librations, equation (12), may now be calculated as follows. The relation between $\{z_i\}$ and $\{Z_i\}$ was derived earlier, viz.

$$\{z_i\} = [RM][EC]^T[P][T]^T\{Z_i\}. \tag{41}$$

Comparing equation (39) with equation (41) provides the result

$$[RM][EC]^T[P][T]^T = [C(\beta''')]$$

or

$$[RM] = [C(\beta''')][T][P]^T[EC]. \tag{42}$$

Now $[C(\beta''')]$ is available from RAI9S, $[T]$ was calculated above, and $[P]$ and $[EC]$ are calculated as they are in EARRO.

The matrix $[PP]$ defined by $[PP] \equiv [RM]^T$ is calculated in RIGEM, viz.

$$[PP] = [EC]^T \; [P] \; [T]^T \; [C\;(\beta''')]^T \; . \tag{43}$$

The elements of [PP] (or [RM]) are functions of θ, $\tilde{\phi}$, ψ, the Euler angles locating the motion with respect to the ecliptic and mean equinox of date system.

Accordingly,

$$\left.\begin{aligned} \tilde{\phi} &= \tan^{-1}\;(-PP\;(3,1)/-PP\;(3,2)) \\ \theta &= \tan^{-1}\;(\sqrt{PP\;(3,1)^2 + PP\;(3,2)^2}/PP\;(3,3)) \\ \psi &= \tan^{-1}\;(-PP\;(1,3)/PP\;(2,3)) \end{aligned}\right\} \; . \tag{44}$$

Knowing these angles the physical librations are

$$\left.\begin{aligned} \rho &= \theta - I \\ \sigma &= \psi - \Omega \\ \tau &= \tilde{\phi} - \Pi - ☾ + \psi \end{aligned}\right\} \; . \tag{45}$$

Finally, the Euler parameter constraints for the Moon are calculated, viz.

$$\left.\begin{aligned} &\Sigma\;\beta_i \\ &\Sigma\;\beta_i\;\dot{\beta}_i \\ &\Sigma\;\beta_i\;\ddot{\beta}_i - \Sigma\;\dot{\beta}_i^{\,2} \end{aligned}\right\} \; . \tag{46}$$

Subroutine PROB. This subroutine provides all initial conditions and constants for RIGEM. These values are also printed out by PROB for reference. Initial conditions for the translational motion and mass parameters are currently being taken from Table 10 of Reference [6]. Initial Euler parameters and rates are taken from ANEAMO.

If the multi-case option is selected, then a branch to statement 12 occurs in this subroutine. The initial values of the time, integration variables and their rates are reset automatically and whatever changes are desired in the parameters must be read in at this point.

Subroutine FORCE. This subroutine calculates the accelerations required by subroutine RAI9S. The calculations are made in the sequence shown in Table 9.

Table 9. Calculation Sequence in Subroutine FORCE.

Calculations	Equations in Reference [1]
A. F (4) - F (33) N Particle Accelerations on all planets and Moon.	10
B. F (1) - F (3) Accelerations on Sun.	= 0 in Version 01
C. F (34) - F (37) Euler parameter accelerations $\{\ddot{\beta}_i'\}$ of Earth.	31
D. F (38) - F (41) Euler parameter accelerations $\{\ddot{\beta}_i'''\}$ of Moon.	50

Basically (Version 01) subroutine FORCE calculates

i) N Body gravitational forces on all bodies treated as particles based on equations (10) and (12) of Reference [1].

ii) Torques on the triaxial Earth due to a point mass Moon are given in equation (98) of Reference [1]. Note: Torques due to a point mass Sun are not included in this version.

iii) Torques on the triaxial Moon due to a point mass Earth and a point mass Sun given by equations (84) and (91) of Reference [1].

All geometrical equations and transformation equations are given in equations (15) - (55) of Reference [1]; except those providing the location of the Sun with respect to frame $\{z_i\}$. They will be developed in the next paragraphs.

The torque components on the Moon due to a point - mass Sun are

$$
\begin{aligned}
Mz_1 &= 3GM_{\odot}\ \alpha\ m_{\odot}\ n_{\odot}\ /r^3_{\odot☾} \\
Mz_2 &= -3GM_{\odot}\ \beta\ \ell_{\odot}\ n_{\odot}\ /r^3_{\odot☾} \\
Mz_3 &= 3GM_{\odot}\ \gamma\ \ell_{\odot}\ m_{\odot}\ /r^3_{\odot☾}
\end{aligned}
\qquad (47)
$$

where $\ell_\odot$, $m_\odot$, $n_\odot$ are the direction cosines of the Sun with respect to the $\{z_i\}$ frame. RIGEM provides the direction cosines of the Moon with respect to the $\{X_i'\}$ frame, viz.

$$\left\{\frac{X_i'}{r_{\odot☾}}\right\} = \frac{x\ (12 + i)}{r_{\odot☾}} \tag{48}$$

The direction cosines of the Sun with respect to a frame $\{X_i'^T\}$ at the Moon are the negative of the ratios given in equations (48).

The direction cosines of the Moon with respect to $\{z_i\}$ can now be formed since the rotation matrices $[T]$ and $[C\ (\beta''')]$ are available (see equations (39) and (15)):

$$\{z_i\} = [C\ (\beta''')]\ \{Z_i\}$$

$$\{Z_i\} = [T]\ \{X_i'\}\ .$$

Accordingly,

$$\left\{\frac{z_i}{r_{\odot☾}}\right\} = -[C\ (\beta''')]\ [T]\ \left\{\frac{X_i'}{r_{\odot☾}}\right\} \tag{49}$$

Subroutine FORCE also calls subroutine FORTOR to provide the remaining forces and torques that are modeled.

Subroutine FORTOR. This subroutine calculates:

i) Force on Earth other than N particle force.

ii) Force on Moon other than N particle force.

iii) Torque on Moon other than gravity-gradient effect due to Sun and Earth.

iv) Torque on Earth other than gravity-gradient effect due to Moon.

Specifically, in version 01, the following forces and torques are calculated:

i) Force on Earth--no additional contribution programmed.

ii) Force on Moon--Force due to Earth's figure and Moon's figure.

iii) Torque exerted on Moon because of lunar higher degree gravity harmonic coefficients and mutual potential terms by an oblate Earth, viz.

a) Torque on Moon due to point mass Earth acting on C_{30}' , C_{31}' , C_{32}' , S_{31}' , S_{32}' , C_{33}' , S_{33}' , C_{41}' , S_{41}' , C_{43}' , S_{42}' , and

b) Torque on Moon due to the interaction of second degree lunar harmonics with Earth oblateness $C_{20}\, C_{20}'$, $C_{20}\, C_{22}'$.

iv) Torque on Earth--no additional contribution programmed.

The above torque expressions were given in equations (85) and (90) of Reference [1].

The force expressions can be obtained from equations (59) - (78) of Reference [1]. They may be put in the following form:

$$F_r = \frac{-3G\, m_4\, m_5\, a^2}{r_{45}^{\,4}} \left[P_{20}\, C_{20} + P_{21}\, (C_{21}\, C\lambda + S_{21}\, S\lambda) + P_{22}\, (C_{22}\, C2\,\lambda + S_{22}\, S2\,\lambda)\right]$$

$$\frac{-3G\, m_4\, m_{50}\, a^{12}}{r_{45}^{\,4}} \left[C_{20}'\, P_{20} + P_{22}\, C_{22}'\, C2\,\lambda\right]$$

$$\frac{1}{r} F_\phi = \frac{G\, m_4\, m_5\, a^2}{r_{45}^{\,4}} \left[C_{20}\, P_{20_\phi} + P_{21_\phi}\, (C_{21}\, C\lambda + S_{21}\, S\lambda) + P_{22_\phi}\, (C_{22}\, C2\,\lambda + S_{22}\, S2\,\lambda)\right]$$

$$\frac{+G\, m_4\, m_5\, a^{12}}{r_{45}^{\,4}} \left[C_{20}'\, P_{20_\phi} + P_{22_\phi}\, C_{22}'\, C2\,\lambda\right]$$

$$\frac{1}{rC\phi} F = \frac{G\, m_4\, m_5\, a^2}{r_{45}^{\,4}\, C\phi} \left[P_{21}\, (-C_{21}\, S\lambda + S_{21}\, C\lambda) + 2P_{22}\, (-C_{22}\, S2\,\lambda + S_{22}\, C2\,\lambda)\right]$$

$$\frac{+G\, m_4\, m_5\, a^{12}}{r_{45}^{\,4}\, C\phi} \left[-2P_{22}\, C_{22}'\, S2\,\lambda\right] \qquad (50)$$

where r_{45} , λ , ϕ are the polar coordinates of the lunar mass center with respect to $\{z_i^T\}$. The quantities P_{20} , P_{21} , P_{22} , P_{20_ϕ} , P_{21_ϕ} , P_{22_ϕ} were defined in Reference [1].

The primed quantities refer to the Moon and the unprimed quantities refer to the Earth. Here,

$$
\begin{aligned}
a^2\ M\ C_{20} &= 1/2\ [A + B + C - 3\ (\alpha''^2\ A + \beta''^2\ B + \gamma''^2\ C)] \\
a^2\ M\ C_{21} &= \alpha\alpha''\ A + \beta\beta''\ B + \gamma\gamma''\ C \\
a^2\ M\ S_{21} &= \alpha'\ \alpha''\ A + \beta'\ \beta''\ B + \gamma'\ \gamma''\ C \\
4a^2\ M\ C_{22} &= A\ (\alpha'^2 - \alpha^2) + B\ (\beta'^2 - \beta^2) + C\ (\gamma'^2 - \gamma^2) \\
2a^2\ M\ S_{22} &= \alpha\alpha'\ A + \beta\beta'\ B + \gamma\gamma'\ C
\end{aligned}
\qquad (51)
$$

where the relative orientation of the Earth and the Moon is given by the $[\ell]$ matrix defined by

$$\{y_i\} = [\ell]\ \{z_i\} \qquad (52)$$

where

$$[\ell] = \begin{bmatrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{bmatrix} .$$

Now, EARRO provides (equation (21))

$$\{y_i\} = [S]\ [N]\ [P]\ \{X_i'\}$$

and RIGEM provides (equations (39) and (15))

$$\{z_i\} = [C\ (\beta''')]\ [T]\ \{X_i'\}$$

so that

$$[\ell] = [S]\ [N]\ [P]\ [T]^T\ [C\ (\beta''')]^T . \qquad (53)$$

The torques are programmed directly from equations (85), (89), (90). Note that the following simplifications have been made in the above torques as programmed in Version 01:

i) In the mutual potential terms, only those multiplied by C_{20} for the Earth are retained,

ii) In the fourth degree terms only those factored by C_{41}' , C_{43}' , S_{42}' , and S_{44}' have been retained since these produce the most significant effects in the librations, [16].

The direction cosines of the Earth with respect to the $\{z_i\}$ frame as required by equations (89) and (90) are immediately available from the $[C\ (\beta''')]$ matrix in RIGEM. The polar coordinates $(r\ ,\ \hat{\phi}\ ,\ \hat{\lambda})$ of the lunar mass center with respect to $\{y_i\}$ are required in the mutual potential terms. These may be found as follows.

Equation (21) provides

$$\{y_i\} = [S]\ [N]\ [P]\ \{X_i'\}$$

or

$$\begin{Bmatrix} C\hat{\phi}\ C\hat{\lambda} \\ C\hat{\phi}\ S\hat{\lambda} \\ S\hat{\phi} \end{Bmatrix} = [S]\ [N]\ [P] = \begin{Bmatrix} C\phi\ C\lambda \\ C\phi\ S\lambda \\ S\phi \end{Bmatrix} \tag{54}$$

The cosines $C\phi\ C\lambda$ etc. are available from RIGEM and $[S]\ [N]\ [P]$ is available from EARRO.

An equivalent but simpler calculation of the effect due to the Earth's figure is found by referring these calculations to the $\{y_i\}$ frame initially. Thus,

$$\{F_{y_i}\} = -\ (3/2)\ \frac{G\ m_4\ m_5\ a^2}{r_{45}{}^4}\ C_{20} \begin{Bmatrix} (5\ \sin^2\phi - 1)\ C\phi'\ C\lambda' \\ (5\ \sin^2\phi - 1)\ C\phi'\ S\lambda' \\ (5\ \sin^2\phi - 3)\ S\phi' \end{Bmatrix} \tag{55}$$

$$\{F_{X_i}\} = [R]^T\ \{F_{y_i}\}\ . \tag{56}$$

In these equations, ϕ' , λ' are referred to $\{y_i\}$ and C_{20} is a constant.

<u>Verification of RIGEM.</u> The translational motion of the centers of mass of the planets and the Moon have been compared with the results given in Reference

[6]. Tables 10 and 11 present the results of this comparison. The comparison was made at J. D. 2442000.5 after an 800-day integration. Table 10 provides the comparison results for all planets using the 11th order option (NOR = 11) in RAI9s. The relatively large errors for the inner planets are due to the exclusion of relativity in this model. Table 11 presents comparisons for the Earth and Moon only for several RAI9S orders. Note that 22 is an accuracy parameter as discussed in Reference [7].

The coupled rotational-translational portion of RIGEM has been verified by fitting the output of RIGEM to the output of ANEAMO as regards the physical librations in node, inclination, and longitude. Details of this comparison are presented in the next section.

III. PARAMETER ESTIMATION METHOD AND PROGRAM ESTEM

In order to verify the numerically integrated lunar librations produced by RIGEM and the analytic librations calculated by ANEAMO in subroutine LURE, a comparison of these two approaches has been undertaken. Also, the eventual use of the unified model in the reduction of LURE data will necessitate a comparison of the observations with the model.

The above comparisons can only be made if the proper set of initial conditions and model parameters is used in the comparison. The numerically integrated librations comprise the model in this study. For a comparison of the model with the analytic theory a set of initial Euler parameters and rates might be taken from ANEAMO since these quantities are calculated there. It has been determined that these parameters and rates are not accurate enough to give the best comparison of the model to the librational theory. There are several reasons for this, among them are the fact that 1) the rates are generated by an approximate numerical method, i.e., by differentiation of a cubic spline fit to the parameter values and 2) the calculation of the Euler parameters themselves has approximations since the calculation makes use of the truncated form of the lunar theory.

It was therefore decided to use a traditional iterative weighted least squares estimation of the initial state and the parameters α , β , γ to insure a "best" fit of the model to the analytical theory.

A complication arose in this process however since the initial Euler parameters and the quantities α , β , γ are not all independent. An iterative

Table 10. Comparison of RIGEM with Reference [6].

Planet	$\lvert\Delta x\rvert$ (Av) $\lvert\Delta \dot{x}\rvert$ (Av/day)	$\lvert\Delta Y\rvert$ $\lvert\Delta \dot{Y}\rvert$	$\lvert\Delta Z\rvert$ $\lvert\Delta \dot{Z}\rvert$
Mercury	2×10^{-5} 5×10^{-7}	4×10^{-6} 1.3×10^{-6}	3×10^{-6} 6×10^{-7}
Venus	1×10^{-10} 1×10^{-10}	3×10^{-9} 7×10^{-11}	2×10^{-9} 4×10^{-11}
Earth	4×10^{-9} 5×10^{-11}	3×10^{-9} 6×10^{-11}	1×10^{-9} 1×10^{-11}
Moon	4×10^{-10} 2×10^{-10}	3×10^{-9} 1×10^{-9}	8×10^{-10} 6×10^{-10}
Mars	2×10^{-8} 2×10^{-10}	5×10^{-8} 4×10^{-10}	2×10^{-8} 2×10^{-10}
Jupiter	4×10^{-11} 1×10^{-12}	2×10^{-10} 6×10^{-14}	7×10^{-11} 2×10^{-14}
Saturn	1×10^{-11} 9×10^{-14}	7×10^{-11} 1×10^{-13}	4×10^{-11} 2×10^{-14}
Uranus	1×10^{-10} 1×10^{-13}	3×10^{-11} 2×10^{-13}	2×10^{-11} 1×10^{-13}
Neptune	5×10^{-11} 9×10^{-14}	1×10^{-10} 2×10^{-13}	6×10^{-11} 1×10^{-13}
Pluto	9×10^{-11} 1×10^{-13}	1×10^{-11} 2×10^{-13}	3×10^{-11} 1×10^{-13}

1) RAI9S used with NOR = 11 and LL = 8.

2) Not all forces in Reference [6] were modeled--only those discussed in regard to subroutines FORCE and FORTOR.

Table 11. Comparison of RIGEM with Reference [6] (Continued).

Planet	Δx (Av)	ΔY (Av)	ΔZ (Av)	Conditions
Earth	-1.81×10^{-6}	8.66×10^{-7}	3.49×10^{-7}	NOR = 7 , LL = 5
Moon	2.75×10^{-5}	2.27×10^{-6}	3.90×10^{-6}	OMPL (1) = 1
				No Earth or Lunar figure
Earth	-1.46×10^{-6}	8.83×10^{-7}	3.9×10^{-7}	NOR = 15 , LL = 10
Moon	-1.02×10^{-6}	8.92×10^{-7}	4.9×10^{-7}	OMPL (1) =.1
				No Earth or Lunar figure
Earth	-0.1×10^{-8}	-2.1×10^{-9}	2.2×10^{-9}	NOR = 7 , LL = 10
Moon	4.7×10^{-7}	6.6×10^{-9}	1.7×10^{-7}	All Planets
				No Earth or Lunar figure
Earth	-3.87×10^{-9}	-2.69×10^{-9}	-4.8×10^{-10}	NOR = 15 , LL = 10
Moon	1.3×10^{-8}	-1.95×10^{-9}	1.44×10^{-9}	All Planets
				Earth figure effect on Lunar orbit
Earth	-3.72×10^{-9}	-2.6×10^{-9}	-1.4×10^{-9}	NOR = 11 , LL = 8
Moon	4.4×10^{-10}	-2.5×10^{-9}	-7.7×10^{-10}	All Planets
				Both Earth and Lunar figure effect on Lunar orbit

weighted least squares (IWLS) method was therefore programmed accounting for the fact that exact constraints must be satisfied between certain estimated variables.

The Euler parameters and rates generated in ANEAMO were utilized as initial guesses to the IWLS process.

A. Iterative Weighted Least Squares with Constraints

Reference [17] provides a formulation of the iterative weighted least squares method when exact constraints are present. A simplified version of that formulation is presented here.

The vector of observations $\vec{y}_o$ can be related to the vector of theoretically calculated values from the model by

$$\vec{y}_o = \vec{y}_C (\vec{x}) + \vec{\varepsilon} \tag{57}$$

where x is a set of parameters and initial conditions and $\vec{\varepsilon}$ is a vector of measurement errors. In this case, the $\vec{y}_C$ is nonlinear in the parameters x and a linearization is made about a nominal set of parameters x^0, viz.

$$\vec{y}_o = \vec{y}_C (\vec{x}^0) + A (\vec{x} - \vec{x}^0) + \vec{\varepsilon} \tag{58}$$

where

$$A_{ij} = \left[\frac{\partial y_{C_i}}{\partial x_j}\right]_{\vec{x}^0} .$$

The traditional derivation of the best estimate, $\hat{\vec{x}}$, to $\vec{x}$ so that ε^2 is minimized in a weighted least squares sense is given by

$$\hat{x} = \vec{x}^0 + (A^T WA)^{-1} A^T W (\vec{y}_o - \vec{y}_C (\vec{x}^0)) , \tag{59}$$

where

$(\ )^T$ is the transposed matrix

$(\ )^{-1}$ is the inverse matrix

W is the weighting matrix.

Also, if $\vec{\varepsilon}$ are samples of zero-mean Gaussian independent random variables and if each observation is weighted with its associated error variance the covariance matrix of the error in x is

$$P = (A^T WA)^{-1} ,$$

the standard error of the estimate of x_i is

$$\sigma_i = \sqrt{P_{ii}} ,$$

and the correlation coefficient between errors in estimates of x_i and x_j is

$$\rho_{ij} = \frac{P_{ij}}{\sqrt{P_{ii}} \sqrt{P_{jj}}} ,$$

as shown in Reference [18].

The above development is modified if the parameters are not all independent. Reference [17] provides the following algorithm in that case:

i) Define

$$\{q\} = \left\{ \frac{x}{S} \right\}$$

where x is a set of "solve-for" parameters and S is a set of "exactly constrained" parameters.

ii) Define the exact constraints by

$$f_i \; (x, S, N_i) = 0 \qquad (i = 1, \ldots, n)$$

where N_i is a set of n constants.

iii) Designate one parameter from each exact constraint as a constrained parameter and solve for it as follows:

$$\{S\} = \begin{Bmatrix} S_1 \ (x) \\ \cdot \\ \cdot \\ \cdot \\ S_n \ (x) \end{Bmatrix}$$

and determine the following partial derivatives

$$[S_x] = \begin{bmatrix} \frac{\partial S_1}{\partial x_1} & \cdot \ \cdot \ \cdot \ \cdot & \frac{\partial S_1}{\partial x_n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ \frac{\partial S_n}{\partial x_1} & & \frac{\partial S_n}{\partial x_n} \end{bmatrix}$$

iv) Form the residual (observed-calculated) vector

$$\{R\} = \begin{Bmatrix} \hat{Z} - Z \\ \tilde{x} - x \end{Bmatrix}$$

where

$\hat{Z}$ are observables

Z are computed observables

$\tilde{x}$ are a priori parameter estimates

x are estimated values.

v) Form the A matrix

$$[A] = \left[A_x \ \vdots \ A_S\right] = \left[\frac{\partial Z}{\partial x}\bigg|_{S \text{ fixed}} \ \vdots \ \frac{\partial Z}{\partial S}\right]$$

where

$$\frac{\partial Z}{\partial S} = A_x + A_S \, S_x$$

vi) Define

$$J = (A_x + A_S \, S_x)^T \, W \, (A_x + A_S \, S_x)$$

where W is a weighting matrix

vii) Then, using a Newton-Raphson integration where $x^{(n)}$ is the nth estimate of $\vec{x}$ and $x^{(n+1)}$ is the (n + 1)st estimate, there is obtained

$$x^{(n+1)} = x^{(n)} + J \quad [(A_x + A_S \, S_x)^T \, W \, (Z - Z \, (x^{(n)})] \tag{60}$$

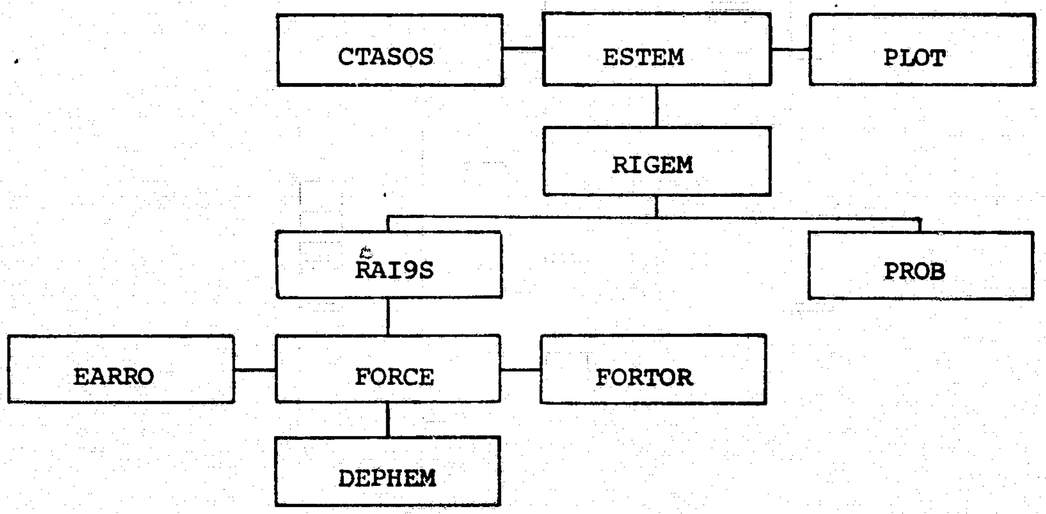
$$S_i^{(n+1)} = S_i \, (x^{(n+1)}) \, . \tag{61}$$

B. Program ESTEM (Version 01)

Program ESTEM was prepared to fit the output from RIGEM to that of ANEAMO by adjusting initial conditions and certain physical parameters in an iterative weighted least squares sense.

Some general features of ESTEM are discussed below.

Program ESTEM links several subprograms as follows:



Subroutines CTASOS and PLOT are LaRc library subroutines for inversion of matrices and plotting. The other subroutines have been discussed previously.

Subroutine FORCE and FORTOR have been modified for use in ESTEM from the version described earlier in this report. The changes in summary form are:

1. The segment that calculates forces on the Sun has been removed.

2. An option has been included that allows the integration of only the Earth and Moon's translational motion (ϕMPL (1) = 2).

3. The capability of reading the JPL ephemeris tape DE69 has been added for those planets not integrated.

4. The segment that calculates the rotational motion of the Earth has been removed.

5. Relativity perturbations on the planet's orbits have been included using the modified one-body Eddington/Roberson equations of [20].

6. The following forces and torques have been added:

i) Lunar torques due to Earth acting on all remaining 4th degree lunar harmonics.

ii) Lunar torques due to interaction of C_{22} with all second degree lunar harmonics.

iii) Acceleration of lunar mass center due to Sun/Earth figure interaction and to Sun/Moon figure interaction.

iv) Acceleration of Earth mass center due to Earth and lunar figures, due to Sun/Earth figure interaction and due to Sun/Moon figure interactions.

The derivation for items i) and ii) follows that given in [1] while the derivation of items iii) and iv) follows [6].

This program basically performs a pre-set number of iterations based on equations (58) and (59). The final estimate of the parameters is used with the model to compute a final set of residuals which are then plotted.

More explicitly,

$$\{x\} = \begin{Bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \dot{\beta}_0 \\ \dot{\beta}_1 \\ \dot{\beta}_2 \\ \beta \\ \gamma \end{Bmatrix} , \qquad \{S\} = \begin{Bmatrix} \beta_3 \\ \dot{\beta}_3 \\ \alpha \end{Bmatrix} , \tag{62}$$

$$\left. \begin{aligned} f_1 &= 1 - \beta_0{}^2 - \beta_1{}^2 - \beta_2{}^2 - \beta_3{}^2 = 0 \\ f_2 &= \beta_0\ \dot{\beta}_0 + \beta_1\ \dot{\beta}_1 + \beta_2\ \dot{\beta}_2 + \beta_3\ \dot{\beta}_3 = 0 \\ f_3 &= \alpha\ (1 - \beta\gamma) - \beta + \gamma = 0 \end{aligned} \right\} , \tag{63}$$

$$\left. \begin{aligned} S_1 &= \beta_3 = \pm\sqrt{1 - \beta_0{}^2 - \beta_1{}^2 - \beta_2{}^2} \\ S_2 &= \dot{\beta}_3 = -\frac{1}{\beta_3}\ (\beta_0\ \dot{\beta}_0 + \beta_1\ \dot{\beta}_1 + \beta_2\ \dot{\beta}_2) \\ S_3 &= \frac{\beta - \gamma}{1 - \beta\gamma} \end{aligned} \right\} \tag{64}$$

$$[S_x] = \begin{bmatrix} \pm\beta_0/S_1 & \pm\beta_1/S_1 & \pm\beta_2/S_1 & 0 & 0 & 0 & 0 & 0 \\ -\dot{\beta}_0/\beta_3 & -\dot{\beta}_1/\beta_3 & -\dot{\beta}_2/\beta_3 & -\beta_0/\beta_3 & -\beta_0/\beta_3 & -\beta_2/\beta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \kappa_1 & \kappa_2 \end{bmatrix} , \tag{65}$$

$$\begin{aligned} \kappa_1 &= (1 - \gamma^2)/(1 - \beta\gamma)^2 \\ \kappa_2 &= (\beta^2 - 1)/(1 - \beta\gamma)^2 \end{aligned} \tag{66}$$

The weighting matrix is taken to be the identity matrix and the a priori parameter estimates and estimated values are set equal to zero in $\{R\}$.

The matrix of observables is

$$
\{\hat{Z}\} = \begin{Bmatrix} \rho_j \\ \circ \\ \circ \\ \circ \\ \sigma_j \\ \circ \\ \circ \\ \circ \\ \tau_j \\ \circ \\ \circ \\ \circ \end{Bmatrix} \qquad (j = 1, \ldots, N)
$$

where ρ, σ, τ are the physical librations in inclination, node, and longitude and N is the number of observations and the matrix $[A]$ is generated by the "variant trajectory" method.

In this application, the $\{\hat{Z}\}$ observables are calculated from Eckhardt's theory as evaluated in ANEAMO. The calculated values $\{Z\}$ are calculated in RIGEM.

The model used is summarized below:

i) $$m_i \ddot{\vec{\rho}}_i = \vec{\nabla}_i U + \vec{F}_i \qquad (i = 1, 11) \tag{67}$$

ii)
$$
\begin{aligned}
\{\ddot{\beta}'''\} = \frac{1}{2} [\dot{\beta}'''] & \left(\begin{Bmatrix} 0 \\ \omega_1' \\ \omega_2' \\ \omega_3' \end{Bmatrix} - [C(\beta''')]_A \begin{Bmatrix} 0 \\ -\dot{\lambda} S\phi \\ \dot{\phi} \\ \dot{\lambda} C\phi \end{Bmatrix} \right) \\
& + \frac{1}{2} [\beta'''] \left(\begin{Bmatrix} 0 \\ \dot{\omega}_1' \\ \dot{\omega}_2' \\ \dot{\omega}_3' \end{Bmatrix} - [C(\beta''')]_A \frac{d}{dt} \begin{Bmatrix} 0 \\ -\dot{\lambda} S\phi \\ \dot{\phi} \\ \dot{\lambda} C\phi \end{Bmatrix} \right. \\
& \left. - \frac{d}{dt} [C(\beta''')]_A \begin{Bmatrix} 0 \\ -\dot{\lambda} S\phi \\ \dot{\phi} \\ \dot{\lambda} C\phi \end{Bmatrix} \right)
\end{aligned} \tag{68}
$$

iii) $[PP] = [EC]^T \; [P] \; [T]^T \; [C]^T$ (69)

iv)
$$\left.\begin{aligned}\tilde{\phi} &= \tan^{-1} \left(-PP\,(3,1)/-PP\,(3,2)\right)\\ \theta &= \tan^{-1} \left(\sqrt{PP\,(3,1)^2 + PP\,(3,2)^2}/PP\,(3,3)\right)\\ \psi &= \tan^{-1} \left(-PP\,(1,3)/PP\,(2,3)\right)\end{aligned}\right\} \quad (70)$$

v)
$$\Omega = a_\Omega + b_\Omega t + c_\Omega t \quad (71)$$
$$☾ = a_☾ + b_☾ t + c_☾ t^2$$

vi)
$$\rho = \theta \dot{-} I$$
$$\sigma = \psi - \Omega \quad (72)$$
$$\tau = \tilde{\phi} - \Pi - ☾ + \psi$$

In the above, equations (65) and (66) are derived in Reference [1], equations (67) and (68) were discussed earlier as equations (43) and (44). Equations (69) are available in Reference [2], and equation (70) was discussed earlier as equation (45).

C. A Verification Run

The results of the verification run are shown in Figure 3. There the residuals in ρ , $I\sigma$, and τ in arc seconds are shown versus time in days past the epoch of J. D. 2441200.5. The fit was made over 1100 days.

The maximum residuals in ρ and $I\sigma$ are about $\pm 3''$ and those in τ are about $\pm 10''$. This plot indicates that no gross errors exist but some subtle inconsistencies still exist in the comparison. The residuals in all angles should be periodic with maximum amplitudes of $<1''\!.5$. Work is continuing on improvement of these residuals as well as a comparison with a more extensive version of the theory (Eckhardt's 300 series).

The following conditions applied for the comparison:

ESTEM:

i) Initial conditions for planetary motions per Reference [6].

ii) Planetary masses per Reference [6].

iii) Lunar nominal Euler parameters and rates

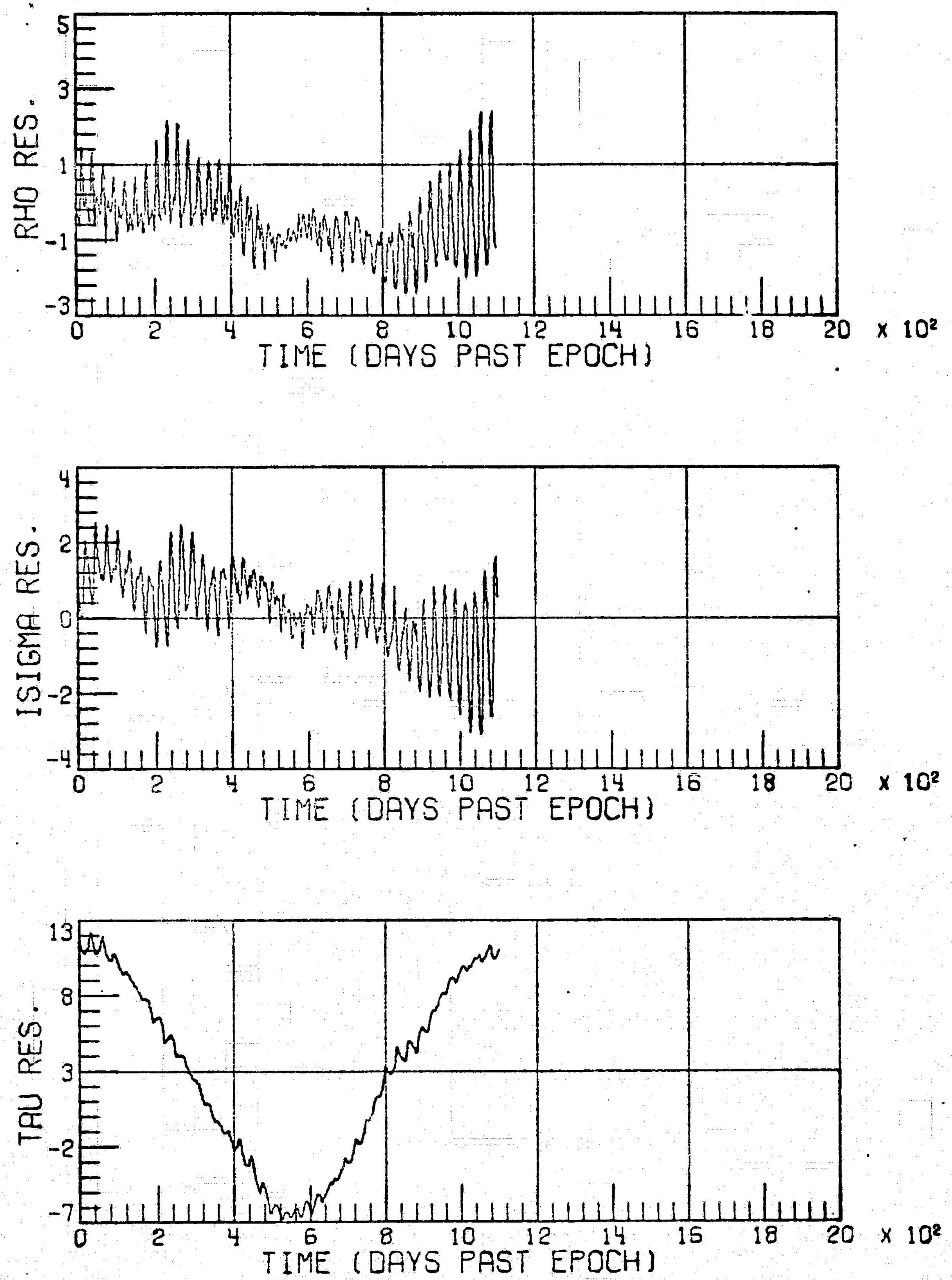


Figure 3. Residuals from Comparison of Physical Librations Using ESTEM with Eckhardt's Theory.

$\beta_0 = -.9805662538581$ $\dot{\beta}_0 = 1.872186798183^{-3}$

$\beta_1 = .1923761339498$ $\dot{\beta}_1 = 7.693100373778^{-3}$

$\beta_2 = 3.629025185335^{-2}$ $\dot{\beta}_2 = 1.478710744051^{-2}$

$\beta_3 = -1.281649334705^{-2}$ $\dot{\beta}_3 = 1.410632074475^{-2}$

iv) $\left.\begin{array}{l} \alpha = 4.023^{-4} \\ \beta = 6.293^{-4} \\ \gamma = 2.27^{-4} \end{array}\right\}$ lunar inertia ratios.

v) Sampling interval = 3 days.

vi) Integration order = 11 (LL = 8) in RAI9S.

vii) Only second degree lunar harmonics considered.

viii) $C_{20} = -1.082637^{-3}$ Earth Oblateness.

ix) Speed of light = 299792.5 km/sec.

x) Relativity perturbations calculated using Eddington/Robertson form, Reference [20].

xi) No tidal coupling effect on Earth and lunar orbits.

xii) Earth and Moon integrated with remaining planetary motions read from JPL DE69 ephemeris tape.

ANEAMO:

i) Eckhardt $j = 3$ model with coefficients as listed in Appendix A.

IV. FUTURE WORK

The next phases of work under NSG-1152 will involve the following:

1. Development of observational equations (laser ranging data) and partial derivatives;

2. Adopting a solution parameter set to be estimated from data;

3. General optimization in terms of running time and storage requirements of all programs; and

4. Development of variational equations for use in parameter estimation process.

To implement the above a series of computer programs based on ANEAMO, RIGEM, and ESTEM is envisional. Collectively these programs are referred to as EMSYS.

The individual programs will have the following capabilities:

EMDYN. i) Numerically integrate all translational, rotational and variational equations of motion from a specified set of initial conditions and parameters; ii) Calculate numerical partials of some quantities if necessary; and iii) Output "ephemeris" of all quantities on tape 10.

EMOBS. i) Read LURE data tape; ii) Apply any necessary corrections to the data; and iii) Write resulting observations on tape 11.

EMNORM. i) Reads tapes 10 and 11; ii) Interpolates tape 10 at observation times; iii) Computes relativistic time delay; iv) Calculates all partial derivatives using analytical results and data from the variational equations as supplied on tape 10; v) Forms the normal equations; and vi) Outputs residuals and normal equations on tape 12.

EMEST. i) Reads tape 12; ii) Solves normal equations for best estimate of parameters; iii) Outputs useful information in printed and plotted form; and iv) Computes covariance matrix and other statistical quantities.

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APPENDIX A

ECKHARDT'S LIBRATION THEORY

(J = 3)

ℓ	ℓ'	F	D	τ	Iσ	ρ
0	0	0	2	- 0.43	0	0
0	0	2	-2	1.63	- 3.2	- 3.2
0	1	0	0	90.71	0	0
1	-1	0	-1	- 1.2	0	0
1	0	-2	0	- 0.48	- 24.46	24.52
1	0	0	-2	4.11	2.5	- 1.95
1	0	0	-1	- 3.46	0	0
1	0	0	0	-16.69	-101.20	-98.97
2	-2	0	-2	0.4	0	0
2	-1	0	-2	0.96	0	0
2	0	-2	0	33.5	0	0
2	0	0	-2	9.87	0	0
2	0	0	0	- 0.4	0.9	0
0	0	2	0	0	- 10.6	-11
1	0	2	-2	0	0.55	0.61
1	0	2	0	0	- 0.85	- 0.65

(I = 5550.2")

APPENDIX B

ERRATA FOR FIRST
SEMIANNUAL PROGRESS REPORT
ON NSG-1152

Corrected Pages
from Reference [1] Follow

Inverting these expressions provides

$$r = \sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2}$$

$$\lambda = \tan^{-1} (\Delta_2/\Delta_1) \quad , \qquad \text{and} \tag{41}$$

$$\phi = \tan^{-1} (\Delta_3/\sqrt{\Delta_1^2 + \Delta_2^2}) \quad .$$

Since the unit vectors $\vec{k}_i$ are related to those of the spherical polar system by

$$\begin{aligned} \vec{k}_1 &= -\vec{\lambda}_r \\ \vec{k}_2 &= -\vec{\lambda}_\lambda \\ \vec{k}_3 &= \vec{\lambda}_\phi \quad , \end{aligned} \tag{42}$$

the inertial angular velocity of the axes $\{Z_i\}$ can be written as

$$\vec{\omega}_{Z/X'} = \dot{\vec{\lambda}} + \dot{\vec{\phi}} = \dot{\lambda} \, \vec{I}_3' - \dot{\phi} \, \vec{\lambda}_\lambda \quad . \tag{43}$$

The above vector may be projected on the $\{Z_i\}$ axes providing

$$\vec{\omega}_{Z/X'} = \dot{\lambda} \left[\cos \phi \, \vec{k}_3 - \sin \phi \, \vec{k}_1\right] + \dot{\phi} \, \vec{k}_2 \quad . \tag{44}$$

The components

$$\begin{aligned} \omega_1' &= -\dot{\lambda} \sin \phi \\ \omega_2' &= \dot{\phi} \\ \omega_3' &= \dot{\lambda} \cos \phi \end{aligned}$$

treated together. Finally the term U_3^I and the remaining terms in $U_4^{\oplus ☾}$ will be treated [see eqs. (86) and (87)].

The reference axes for the second order and coupling terms are $\{y_i\}$. Thus the c'_{ij} and s'_{ij} are functions of the orientation angles. These functions are

$$c'_{20} = \frac{1}{a'^2M'}\left[\frac{I'_{y1y1} + I'_{y2y2}}{2} - I_{y_3y_3}\right]$$

$$= \frac{1}{a'^2M'}\left[\frac{A + B + C}{2} - \frac{3}{2}\,(A'\gamma^2 + B'\gamma'^2 + C'\gamma''^2)\right]$$

$$c'_{21} = \frac{1}{a'^2M'}\, I'_{y1y3}$$

$$= -\frac{1}{a'^2M'}\,[\alpha\gamma A' + \alpha'\gamma'B' + \alpha''\gamma''C']$$

$$s'_{21} = \frac{1}{a'^2M'}\, I'_{y3y2} \tag{80}$$

$$= -\frac{1}{a'^2M'}\,[\gamma\beta A' + \gamma'\beta'B' + \gamma''\beta''C']$$

$$c'_{22} = \frac{1}{4a'^2M'}\Big[A'(\beta^2 - \alpha^2) + B'(\beta'^2 - \alpha'^2)$$

$$+ C'(\beta''^2 - \alpha''^2)\Big]$$

$$s'_{22} = -\frac{1}{4a'^2M'}\,[\alpha\beta A' + \alpha'\beta'B' + \alpha''\beta''C'] \quad .$$

$$M_{z_1} = GMr^{-5}a^2(B' - C')\Big[6P_{40}\{c_{20}(\alpha''\gamma' + \gamma''\alpha')$$

$$+ c_{22}(\beta'\beta'' - \alpha'\alpha'')\} + 3P_{41}\{-c_{22}(\alpha''\gamma' + \gamma''\alpha')c\lambda$$

$$- s\lambda(\gamma'\beta'' + \gamma''\beta')(c_{20} + c_{22})\}$$

$$- \frac{P_{42}}{2}c_{20}\{c2\lambda(\alpha'\alpha'' - \beta'\beta'') + s2\lambda(\alpha''\beta' + \alpha'\beta'')\}$$

$$- 3P_{42}c_{22}c2\lambda\gamma'\gamma'' - \frac{P_{43}}{2}c_{22}\{c3\lambda(\alpha''\gamma' + \alpha'\gamma'')$$

$$+ s3\lambda(\beta''\gamma' + \beta'\gamma'')\} - \frac{P_{44}}{2}c_{22}\{c4\lambda(\beta'\beta'' - \alpha'\alpha'')$$

$$+ s4\lambda(\alpha''\beta' + \beta''\alpha')\}\Big]$$

$$M_{z_2} = GMr^{-5}a^2(C' - A')\Big[6P_{40}\{c_{20}(\gamma''\alpha + \gamma\alpha'')$$

$$+ c_{22}(\beta\beta'' - \alpha\alpha'')\} + 3P_{41}\{-c_{22}(\alpha\gamma'' + \gamma\alpha'')c\lambda$$

$$- s\lambda(c_{20} + c_{22})(\beta\gamma'' + \gamma\beta'')\} + \frac{P_{42}}{2}c_{20}\{(\beta\beta''$$

$$- \alpha\alpha'')c2\lambda - (\alpha\beta'' + \beta\alpha'')s2\lambda\} - 3P_{42}c_{22}c2\lambda\gamma\gamma'' \tag{85}$$

$$- \frac{P_{43}}{2}c_{22}\{(\alpha\gamma'' + \gamma\alpha'')c3\lambda + (\beta\gamma'' + \gamma\beta'')s3\lambda\}$$

$$+ \frac{P_{44}}{4}c_{22}\{(\beta\beta'' - \alpha\alpha'')c4\lambda - (\alpha\beta'' + \beta\alpha'')s4\lambda\}\Big]$$

$$M_{z_3} = GMr^{-5}a^2(B' - A')\Big[6P_{40}\{-c_{20}(\alpha'\gamma + \gamma'\alpha)$$

$$+ c_{22}(\alpha\alpha' - \beta\beta')\} + 3P_{41}\{-c_{22}c\lambda(\alpha'\gamma + \gamma'\alpha)$$

$$- (c_{20} + c_{22})s\lambda(\beta'\gamma + \gamma'\beta)\} + \frac{P_{42}}{2}c_{20}\{c2\lambda(\alpha\alpha'$$

$$- \beta\beta') - s2\lambda(\alpha\beta' + \beta\alpha')\} - 3P_{42}c_{22}c2\lambda\gamma\gamma'$$

$$+ \frac{P_{43}}{2}c_{22}\{-(\alpha'\gamma + \gamma'\alpha)c3\lambda - (\beta'\gamma + \gamma'\beta)s3\lambda\}$$

$$+ \frac{P_{44}}{4}c_{22}\{+(\alpha\alpha' - \beta\beta')c4\lambda - s4\lambda(\alpha'\beta + \alpha\beta')\}\Big]$$